MATH2050a Mathematical Analysis I

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Exercise 1 suggested Solution

1. Show that sup{1-1/n: $n \in N\} = 1$.

Solution:

Archimedean property: If $x \in R$, then there exists n_x such that $x < n_x$.

Let S = {1-1/n :$n \in N\}$. Clearly, $0 < 1/n \leq 1$ for all $n \in N$. Multiply the inequality by negative sign. $0 > -1/n \geq -1$ for all $n \in N$. Add 1 for each term in the above inequality.

$1 - 0 > 1 - 1/n \geq 1 - 1$, $\forall n \in N$; $1 > 1 - 1/n \geq 0$, $\forall n \in N$; $0 \leq 1 - 1/n < 1$, $\forall n \in N$.

That is , Thus, 0 is the lower bound of S and 1 is the upper bound of S.

Since, every non-empty set which has an upper bound as well as lower bound also has a supremum and an infimum in R. Therefore, the set S has a supremum.

Moreover, for any $\epsilon > 0$, by Archimedean property, there exists $n \in N$ such that $1/\epsilon < n$, which implies $1/n < \epsilon$. Further, $1 - 1/n > 1 - \epsilon$. Therefore, $1 - \epsilon$ is not an upper bound of the set for each $\epsilon > 0$.

Since, $\epsilon > 0$ is arbitrary, it follows that 1 is the supremum of the set $S = \{1 - 1/n : n \in N\}$. Hence, sup{1-1/n: $n \in N\} = 1$.

4. Let S be a non-empty bounded set in R.

(a) Let $a > 0$, and let $aS := \{as : s \in S\}$. Prove that

inf(aS) = a inf S, sup(aS) = a sup S .

(b) Let $b < 0$, and let $bS := \{bs : s \in S\}$. Prove that inf(bS) = b sup S, sup(bS) = b inf(S) .

Solution:

Here we just prove (a),and consider the supremum of S.

Recollect definition of supremum. If S is bounded above, then a number u is said to be supremum of S if it satisfies the conditions

(1)u is an upper bound of S, and

(2)if v is any upper bound of S, then $u \leq v$.

Assume u is the supremum of S. since $a > 0$, $\forall x \in S$, $ax \leq au$, which satisfies condition(1).

suppose v is any upper bound of aS, since $a > 0$, then v/a is the upper bound of S. And since u is the supremum of S, we have $u \leq v/a$, which implies $au \leq v$. The conditio(2) also holds.

6. Let X be a nonempty set and let $f : X \to R$ have bounded rang in R. If $a \in R$, show that Example 2.4.1(a) implies that

$sup\{a + f(x) : x \in X\} = a + sup\{f(x) : x \in X\}$

Show that we also have

$inf\{a + f(x) : x \in X\} = a + inf\{f(x) : x \in X\}$

Solution:

Here we just prove the first equality. Let X be a nonempty set, and $f : X \to R$ be a bounded function. Let A:= $\{f(x) : x \in X\}$, we see that $\{a + f(x) : x \in X\} = \{a + y : y \in A\}$. Assume u is the supremum of A, similar to exercise 4, we need to verify $a + u$ satisfies the above two conditions.

$\forall y \in A$, since $y \leq u$, we have $a + y \leq a + u$. So $a + u$ is the bound of $\{a + f(x) : x \in X\}$. suppose v is any upper bound of $\{a + Y : y \in A\}$, then $v - a$ is the upper bound of $\{y : y \in A\}$. Since u is the supremum of A, we have $u \leq v - a$, which implies $a + u \leq v$.The conditio(2) also holds.

Hence, $a + u$ is the supremum of $\{a + f(x) : x \in X\}$, we have $sup\{a + f(x) : x \in X\} = a + sup\{f(x) : x \in X\}$.